Joint State Sensing and Communication over Memoryless Multiple Access Channels

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***Abstract*—A memoryless state-dependent multiple access channel (MAC) is considered where two transmitters wish to convey a respective message to a receiver while simultaneously estimating the respective channel state via generalized feedback. The scenario is motivated by a joint radar and communication system where the radar and data applications share the same bandwidth. An achievable capacity-distortion tradeoff region is derived that outperforms a resource-sharing scheme through a binary erasure MAC with binary states.**

I. Introduction

Consider the communication setup depicted in Fig. 1. Two encoders each wish to convey a message to a decoder over a state-dependent multiple access channel (MAC) and simultaneously estimate their state sequence via generalized feedback $Z_{k,i-1}$, $k = 1, 2$, $i = 2, \dots, n$. For simplicity, we assume that at time i the decoder has access to the state $S_i = (S_{1i}, S_{2i})$. The above communication setup is motivated by joint radar and data communications, where radar-equipped transmitters track the state while exchanging data. Most current communication systems build on resource sharing, where the time and frequency resources are divided into either state sensing or communication.

We recently studied a single-user version of this problem in [1]. In this paper, we extend the results to two-user MACs. As in [1], the state information is available at the receiver, which is different from [2] where the state is estimated at the receiver. The main contributions of the paper are:

- an outer bound on the capacity-distortion region that builds on [3];
- an achievable rate-distortion region that builds on [4];
- numerical examples based on a binary erasure MAC.

This paper is organized as follows. Section II describes the model and presents our main results. Section III provides the outer bound and Section IV provides the achievability proof. We consider a binary erasure MAC with binary states in Section V.

II. System Model and Main Results

Consider the channel inputs $X_{ki} \in \mathcal{X}_k$, the channel outputs $Y_i \in \mathcal{Y}$, the feedback channel outputs $Z_{ki} \in \mathcal{Z}$, and channel state $S_i \in \mathcal{S}_1 \times \mathcal{S}_2$, $k = 1, 2$, $i = 1, \dots, n$ linked by a discrete memoryless channel with i.i.d. states. The joint probability distribution of these random variables can be written as

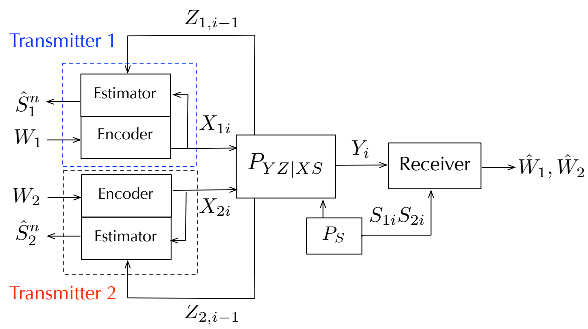

Fig. 1. State-dependent MAC with generalized feedback

$$\prod_{i=1}^n P_S(s_i) P_{YZ_1Z_2|X_1X_2S}(y_i, z_{1i}, z_{2i}|x_{1i}, x_{2i}, s_i) \quad P(x_{1i}|x_1^{i-1}, z_1^{i-1}) P(x_{2i}|x_2^{i-1}, z_2^{i-1}). \tag{1}$$

A $(2^{nR_1}, 2^{nR_2}, n)$ code for the state-dependent discrete memoryless MAC with generalized feedback consists of

- Two message sets $\mathcal{W}_k = [1 : 2^{nR_k}]$ for $k = 1, 2$.
- Encoder k: a function $\phi_{ki} : \mathcal{W}_k \times \mathcal{Z}_k \mapsto \mathcal{X}_k$ that assigns a symbols $x_{ki} = \phi_{ki}(w_k, z_k^{i-1})$ for $i = 1, \dots, n$. For simplicity, we write $x_k^n = \phi_k^n(w_k, z_k^{n-1})$ for the sequence of n encoded symbols.
- Decoder: a function $g : \mathcal{Y}^n \times \mathcal{S}_1^n \times \mathcal{S}_2^n \mapsto \mathcal{W}_1 \times \mathcal{W}_2$ that assigns a message pair $(\hat{w}_1, \hat{w}_2) = g(y^n, s^n)$.
- State estimator k outputs the estimate $\hat{s}_k^n$ as a function of x_k^n and z_k^n. We consider without loss of generality a function $\tilde{\psi}_k^n : \mathcal{X}_k^n \times \mathcal{Z}_k^n \mapsto \mathcal{S}_k^n$ [1, Lemma 2] so that $\hat{s}_k^n = \tilde{\psi}_k^n(x_k^n, z_k^n)$.

The average distortion of estimator k is

$$d_k^{(n)} = \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^n d_k(S_{ki}, \hat{S}_{ki})\right] \tag{2}$$

where $d_k : \mathcal{S}_k \times \hat{\mathcal{S}}_k \mapsto [0, \infty)$ measures the distortion between a state symbol and a reconstruction symbol. We consider bounded distortion functions with $d_{\max} \overset{\Delta}{=} \max_{(k,s,\hat{s})} d_k(s, \hat{s})$. Let the average error probability be $P_e^{(n)}$. We say that

(R_1, R_2, D_1, D_2) is achievable if for all $\epsilon > 0$ there is some n and a $(2^{nR_1}, 2^{nR_2}, n)$ code satisfying $P_e^{(n)} \leq \epsilon$ and $d_k^{(n)} \leq D_k + \epsilon$ for $k = 1, 2$. The capacity region $\mathcal{C}(D_1, D_2)$ is the closure of achievable (R_1, R_2) for specified D_1, D_2.

For our outer bound on $\mathcal{C}(D_1, D_2)$, we consider idealized transmitter estimators $\hat{s}_k = \psi_k^*(x_1, x_2, z_1, z_2)$, $k = 1, 2$, that are aware of x_1, x_2 as well as z_1, z_2. The best such estimators are

$$\psi_k^*(x_1, x_2, z_1, z_2) = \arg \min_{\psi_k : \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{Z}_1 \times \mathcal{Z}_2 \mapsto \mathcal{S}_k} \sum_{s_k \in \mathcal{S}_k} P_{S_k|X_1X_2Z_1Z_2}(s_k|x_1x_2z_1z_2) d_k(s_k, \psi_k(x_1, x_2, z_1, z_2)) \tag{3}$$

for $k = 1, 2$ with the conditional distortions

$$c_k(x_1, x_2) = \mathbb{E}[d_k(s_k, \psi_k^*(x_1, x_2, z_1, z_2))|X_1 = x_1, X_2 = x_2]. \tag{4}$$

The following outer bound extends a bound from [3] to state-dependent MACs with distortion constraints.

Theorem 1. *$\mathcal{C}(D_1, D_2)$ is a subset of the union of (R_1, R_2) satisfying*

$$R_1 \leq I(X_1; Y Z_1 Z_2 | S X_2 T) \tag{5a}$$
$$R_2 \leq I(X_2; Y Z_1 Z_2 | S X_1 T) \tag{5b}$$
$$R_1 + R_2 \leq I(X_1 X_2; Y Z_1 Z_2 | S T) \tag{5c}$$
$$R_1 + R_2 \leq I(X_1 X_2; Y | S) \tag{5d}$$

where $T - SX_1X_2 - YZ_1Z_2$ forms a Markov chain, and we have the dependence balance constraint

$$I(X_1; X_2 | T) \leq I(X_1; X_2 | Z_1 Z_2 T) \tag{6}$$

and the average distortion constraints

$$\mathbb{E}[c_k(X_1, X_2)] \leq D_k, \quad k = 1, 2. \tag{7}$$

It suffices to consider T whose alphabet $\mathcal{T}$ has cardinality $|\mathcal{T}| \leq 7$ (see [9, Appendix B]).

Remark 1. *The result yields a number of special cases studied in the literature. Without distortion constraints and states, the bounds reduce to the ones derived in [3]. For a single user, i.e., X_2 and Z_2 constants, Theorem 1 yields the capacity-distortion tradeoff in [1]. For a special case when the feedback is output feedback $Z_1 = Z_2 = Y$ and we have no distortion constraints, the region reduces to [5, Section VII].*

For our achievable region, we consider an estimator $\underline{\psi}_1^*(x_1, v_2, z_1)$ given by

$$\underline{\psi}_1^*(x_1, v_2, z_1) = \arg \min_{\psi_1 : \mathcal{X}_1 \times \mathcal{V}_2 \times \mathcal{Z}_1 \mapsto \mathcal{S}_1} \sum_{s_1 \in \mathcal{S}_1} P_{S_1|X_1V_2Z_k}(s_k|x_1v_2z_k) d_1(s_1, \psi_1(x_1, v_2, z_k)) \tag{8}$$

yielding the estimation cost as

$$\underline{c}_1(x_1, v_2) = \mathbb{E}[d_1(s_1, \underline{\psi}_1^*(x_1, v_2, z_1))|X_1 = x_1 V_2 = v_2] \tag{9}$$

We define $\underline{\psi}_2^*(v_1, x_2, z_2)$ and $\underline{c}_2(v_1, x_2)$ similarly. The following achievable region is based on [4].

Theorem 2. *$\mathcal{C}(D_1, D_2)$ includes the (R_1, R_2) satisfying*

$$R_1 \leq I(X_1; Y | X_2 V_1 U S) + I(V_1; Z_2 | X_2 U) \tag{10a}$$
$$R_2 \leq I(X_2; Y | X_1 V_2 U S) + I(V_2; Z_1 | X_1 U) \tag{10b}$$
$$R_1 + R_2 \leq \min\{I(X_1 X_2; Y | S), I(X_1 X_2; Y | S V_1 V_2 U) + I(V_1; Z_2 | X_2 U) + I(V_2; Z_1 | X_1 U)\} \tag{10c}$$

where $V_1X_1 - U - V_2X_2$ and $UV_1V_2 - X_1X_2 - YZ_1Z_2$ form Markov chains, and where

$$\mathbb{E}[\underline{c}_1(X_1, V_2)] \leq D_1 \tag{11a}$$
$$\mathbb{E}[\underline{c}_2(V_1, X_2)] \leq D_2. \tag{11b}$$

III. Converse

This section provides a sketch of proof for Theorem 1. Details are provided in [9, Appendix A]. By following the same steps as [3] and [5], we have

$$nR_1 \leq \sum_{i=1}^{n} I(X_{1i}; Y_i Z_i | S_i X_{2i} Z^{i-1}) + n\epsilon \tag{12a}$$
$$nR_2 \leq \sum_{i=1}^{n} I(X_{2i}; Y_i Z_i | S_i X_{1i} Z^{i-1}) + n\epsilon \tag{12b}$$
$$n(R_1 + R_2) \leq \sum_{i=1}^{n} I(X_{1i} X_{2i}; Y_i Z_i | S_i Z^{i-1}) + n\epsilon \tag{12c}$$
$$\sum_{i=1}^{n} I(X_{1i}; X_{2i} | Z_i Z^{i-1}) \leq \sum_{i=1}^{n} I(X_{1i}; X_{2i} | Z^{i-1}). \tag{12d}$$

where we let $Z_i = (Z_{1i}, Z_{2i})$. Next, suppose a genie gives both inputs $X_{1,i}, X_{2,i}$ to both transmitters when estimating $S_{k,i}$ for $j \neq k$. We then have the distortion constraints

$$\frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[c_k(X_{1i}, X_{2i})] \leq D_k + \epsilon, \quad k = 1, 2. \tag{13}$$

Let Q be uniform over $1, 2, \ldots, n$ and independent of all other random variables. Define $T = (Q, Z_1^{Q-1}, Z_2^{Q-1})$, $X_{1Q} = X_1$, and similarly for all other variables. By letting $n \to \infty$, we readily obtain (5a), (5b), (5c), (6) and (7), while (5d) follows from the cut set bound.

IV. Achievability

We use block Markov encoding and backward decoding [4]. Encoder k sends $2(B-1)$ i.i.d. messages $\{w_{k1}(b), w_{k2}(b)\}_{b=1}^{B-1}$ over $n = BN$ channel uses. The messages $w_{k1}(b) \in [1, 2^{NR_{k1}}]$ and $w_{k2}(b) \in [1, 2^{NR_{k2}}]$, $k = 1, 2$, $b = 1, ..., B-1$, are uniformly distributed and mutually independent. By letting $B \to \infty$, we obtain $R_{jk} \frac{B-1}{B} \to R_{jk}$ for any $j, k = 1, 2$. Encoder 1's message w_{12} is decoded by encoder 2, while encoder 2's message w_{21} is decoded by encoder 1 thanks to generalized feedback, yielding encoder cooperation.

a) Codebook Generation: Fix a pmf $P_U(u)\prod_{k=1}^{2} P_{V_k|U}(v_k|u)P_{X_k|V_kU}(x_k|v_k,u)$ and functions $\underline{\psi}_1^*(x_1,v_2,z_1), \underline{\psi}_2^*(v_1,x_2,z_2)$ such that the distortion constraints are satisfied. For each block $b = 1,\dots,B$, we proceed as follows:

- Generate $2^{N(R_{12}+R_{21})}$ sequences $u^N(j_{b-1},k_{b-1})$, $j_{b-1} = 1,...,2^{NR_{12}}$, $k_{b-1} = 1,\dots,2^{NR_{21}}$, each according to $\prod_{i=1}^{N} P_U(u_i)$.
- For each (j_{b-1},k_{b-1}), generate $2^{NR_{12}}$ sequences $v_1^N(j_{b-1},k_{b-1},j'_b)$, $j'_b = 1,\dots,2^{NR_{12}}$, each according to $\prod_{i=1}^{N} P_{V_1|U}(v_{1i}|u_i(j_{b-1},k_{b-1}))$. Similarly generate $v_2^N(j_{b-1},k_{b-1},k'_b)$, $k'_b = 1,\dots,2^{NR_{21}}$.
- For each (j_{b-1},k_{b-1},j'_b), generate $2^{NR_{11}}$ sequences $x_1^N(j_{b-1},k_{b-1},j'_b,l_b)$, $l_b = 1,\dots,2^{NR_{11}}$, each according to $\prod_{i=1}^{N} P_{X_1|UV_1}(x_{1i}|u_i(j_{b-1},k_{b-1}),v_{1i}(j'_b))$. Similarly generate $x_2^N(j_{b-1},k_{b-1},k'_b,m_b)$, $m_b = 1,\dots,2^{NR_{22}}$.

b) Encoding: We set $j_0 = k_0 = 1$ and $l_B = m_B = 1$. At the end of block b, encoder 1 finds an index k'_b such that

$$\big(u^N(\cdot,\cdot)v_1^N(\cdot,\cdot,j'_b), v_2^N(\cdot,\cdot,k'_b), x_1^N(\cdot,\cdot,j'_b,l_b), z_1^N(b)\big) \in \mathcal{T}_\epsilon^N \tag{14}$$

where the first two arguments of each variable are j_{b-1},k_{b-1}[1]. Using this estimate k'_b from block b, encoder 1 transmits $x_1^N(j_b,k'_b,j'_{b+1},l_{b+1})$ in block $b+1$. Similarly, encoder 2 finds an index j'_b such that

$$\big(u^N(\cdot,\cdot), v_1^N(\cdot,\cdot,j'_b), v_2^N(\cdot,\cdot,k'_b), x_2^N(\cdot,\cdot,k'_b,m_b), z_2^N(b)\big) \in \mathcal{T}_\epsilon^N. \tag{15}$$

Using the estimate j'_b from block b, encoder 2 transmits $x_2^N(j'_b,k_b,k'_{b+1},m_{b+1})$ in block $b+1$. Both encoders repeat the same procedure for each b.

c) Decoding: Assuming that (j'_b,k'_b) is decoded correctly in block $b+1$, the decoder finds $(j_{b-1},k_{b-1},l_b,m_b)$ in block b such that $u^N(j_{b-1},k_{b-1})$, $v_1^N(j_{b-1},k_{b-1},j'_b)$, $v_2^N(j_{b-1},k_{b-1},k'_b)$, $x_1^N(j_{b-1},k_{b-1},j'_b,l_b)$, $x_2^N(j_{b-1},k_{b-1},k'_b,m_b)$, $s^N(b), y^N(b)$ are jointly typical. The decoder repeats this step for blocks B to 1.

d) State Estimation: For each block $b = 1,\dots,B$, encoder 1 puts out

$$\hat{s}_1^N(b) = \underline{\psi}_1^*(x_1^N(j_{b-1},k_{b-1},j'_b,l_b), v_2^N(j_{b-1},k_{b-1},k'_b), z_1^N(b))$$

where k'_b is decoded at the end of block b during encoding process. Similarly, encoder 2 lets

$$\hat{s}_2^N(b) = \underline{\psi}_2^*(v_1^N(j_{b-1},k_{b-1},j'_b), x_2^N(j_{b-1},k_{b-1},k'_b,m_b), z_1^N(b))$$

where j'_b is known to encoder 2 from its encoding process.

[1] If there is more than one such index, we select one of these indices uniformly at random. If there is no such index, we choose an index from $\{1,\dots,2^{NR_{21}}\}$ uniformly at random. A similar procedure applies to decoding and shall be omitted.

e) Error Probability: Following the same steps as [4], we can prove that by letting $N \to \infty$, $P_e^{(n)} \to 0$ if the following conditions hold:

$$R_{12} \le I(V_1;Z_2|X_2U) \tag{16a}$$
$$R_{21} \le I(V_2;Z_1|X_1U) \tag{16b}$$
$$R_{11} \le I(X_1;Y|SX_2V_1U) \tag{16c}$$
$$R_{22} \le I(X_2;Y|SX_1V_2U) \tag{16d}$$
$$R_{11}+R_{22} \le I(X_1X_2;Y|SV_1V_2U) \tag{16e}$$
$$R_{12}+R_{21}+R_{11}+R_{22} \le I(X_1X_2;Y|S) \tag{16f}$$

The analysis details are provided in [9, Appendix C]. Applying Fourier-Motzkin elimination, we obtain the desired expressions.

f) Distortion: If there is no decoding error, $(u^N(b)$, $v_1^N(b)$, $v_2^N(b)$, $x_1^N(b)$, $x_2^N(b)$, $y^N(b)$, $s(b))$ are jointly typical for all b. We simplify notation and let $w_k = (w_k(1),\dots,w_k(B-1))$ with $|\mathcal{W}_k| = 2^{N(B-1)(R_{k1}+R_{k2})}$ for $k=1,2$, where $w_k(b)$ denotes $(w_{k1}(b),w_{k2}(b))$. For a given message pair (w_1,w_2), we bound the average distortion for encoder 1.

$$d_1^{(n)}(w_1,w_2) \le P_e^{(n)}(w_1,w_2)d_{\max} + (1-P_e^{(n)}(w_1,w_2))(1+\epsilon)\mathbb{E}\left[\underline{c}_1(X_1,V_2)\right].$$

By averaging over all possible message pairs, we obtain the desired result. The details of the proof are provided in [9, Appendix D].

V. Example

Consider a MAC where the state and channel inputs are binary, $S_k, X_k \in \{0,1\}$ and the channel output is ternary:

$$Y = S_1X_1 + S_2X_2. \tag{17}$$

Consider Hamming distance, i.e., $d(s,\hat{s}) = s \oplus \hat{s}$. For simplicity, we consider output feedback $Z_1 = Z_2 = Y$ and assume that S_1 and S_2 are i.i.d. Bernoulli with parameter $p_s \triangleq \Pr(S=1)$. If $p_s = 1$, then this channel reduces to the binary erasure MAC with feedback, whose capacity region is the Cover-Leung region [6], [7] (see also [8, Chapter 17]).

We compute the optimal estimation cost. The best estimator gives either zero distortion or $\eta = \min\{p_s, 1-p_s\}$ yielding the following cost for encoder 1 (see [9, Appendix E]):

$$c_1(0,0) = \eta P_Y(0), \quad c_1(1,1) = \eta P_Y(1)$$
$$c_1(0,1) = \eta(P_Y(0)+P_Y(1)), \quad c_1(1,0) = 0 \tag{18}$$

A. Proposed Scheme

We characterize an achievable tradeoff between the sum rate and the symmetric distortion of our proposed scheme.

$$X_k = V_k \oplus \Theta_k = U \oplus \Sigma_k \oplus \Theta_k, \quad k=1,2 \tag{19}$$

where $U, \Sigma_1, \Sigma_2, \Theta_1, \Theta_2$ are mutually independent. For the sake of simplicity, we focus on the symmetric rate $R_1 = R_2$ and let U, Σ_k, Θ_k is Bernoulli distributed with parameter p, q, r, respectively, for $k=1,2$.

a) Unconstrained sum rate: We first characterize the unconstrained sum rate without distortion constraints, denoted by $R_{\text{sum-prop}}(\infty)$.

Corollary 1. *The unconstrained sum rate is given by:*

$$R_{\text{sum-prop}}(\infty) = \max_{(p,q,r)} \min\{f_1(p,q,r), f_2(p,q,r)\} \quad (20)$$

with $f_1 = f_{1a} + 2\{f_{1b} - f_{1c}\}$*, where* $f_{1a}, f_{1b}, f_{1c}, f_2$ *are defined in* (21) *by letting* $\kappa = qr + \bar{q}\bar{r}$ *and* $\bar{\kappa} = 1 - \kappa$*.*

The proof is provided in [9, Appendix G].

Remark 2. *For a special case of the erasure MAC with* $p_s = 1$*, the functions* f_1, f_2 *simplifies into:*

$$f_1(p,q,r) = H_3(r^2, 2r\bar{r}, \bar{r}^2) + 2(H_2(\kappa) - H_2(r))$$
$$f_2(p,q,r) = H_3(\bar{p}\kappa^2 + p\bar{\kappa}^2, 2\kappa\bar{\kappa}, p\kappa^2 + \bar{p}\bar{\kappa}^2)$$

It readily follows that f_2 *is maximized by letting* $p = 1/2$*, yielding* $H_2(2\kappa\bar{\kappa}) + \kappa^2 + \bar{\kappa}^2$*. It can be proved that the sum rate is given by choosing* $r = 0$*, yielding*

$$R_{\text{sum-prop}}(\infty) = \max_q \min\{2H_2(q), H_2(2q\bar{q}) + q^2 + \bar{q}^2\}.$$

By choosing $q^* = 0.2377$*, the sum capacity of 1.5822 bit/channel use is achieved [7].*

b) Minimum distortion: The minimum distortion $D_{\min}$ can be obtained by solving the following optimization problem.

$$\min_{p,q,q} \sum_{(x_1,x_2)} P_{X_1,V_2}(x_1,v_2)\underline{c}_1(x_1,v_2) \quad (22)$$

where by letting $\eta_x = \min\{p_s x, 1 - p_s x\}$ the cost function $\underline{c}_1(x_1, v_2)$ is given in [9, Appendix F].

$$\underline{c}_1(0,0) = \eta P_Y(0) + \eta_r P_Y(1), \quad \underline{c}_1(1,1) = P_Y(1)\eta_{\bar{r}}$$
$$\underline{c}_1(0,1) = \eta(P_Y(0) + P_Y(1)), \quad \underline{c}_1(1,0) = P_Y(1)\eta_r \quad (23)$$

The solution of (22) is achieved by choosing $X_1 = X_2 = U$, yielding zero sum rate. With this choice ($q = r = 0$), the estimation cost coincides with the idealized one. Intermediate points between the unconstrained sum rate and the minimum distortion can be evaluated by the parametrized optimization similarly to the single-user case [1].

B. Resource-Sharing

We consider a resource sharing scheme that uses feedback only for state estimation purpose. Then, we can achieve $(D_{\min}, 0)$. The other extreme point is the unconstrained sum rate point without feedback. After some straightforward computation, we obtain:

$$\begin{aligned} R_{\text{sum-no-fb}}(\infty) &= \max_{P_Q P_{X_1|Q} P_{X_2|Q}} H(Y|SQ) \\ &= \max_a 2p_s\bar{p}_s H_2(a) + p_s^2 H_3(a^2, 2a\bar{a}, \bar{a}^2) \\ &= 2p_s\bar{p}_s + \frac{3p_s^2}{2} \end{aligned}$$

where the last equality holds by choosing $a = \frac{1}{2}$. The corresponding distortion is given by a fixed estimator independent of feedback. Namely, we consider $\hat{s}_k = 0$ if $p_s < \frac{1}{2}$ and $\hat{s}_k = 1$ if $p_s \geq \frac{1}{2}$. This yields the distortion of $\eta = \min\{p_s, 1-p_s\}$. In summary, the resource sharing scheme achieves any tradeoff between $(D_{\min}, 0)$ and $(\eta, R_{\text{sum-no-fb}})$.

C. Outer Bound

By applying the upper bound (1) to the binary erasure MAC with binary states, we have

$$R_k \leq H(Y|SX_jT), \ \forall k = 1,2, \forall j \neq k \quad (24a)$$
$$R_1 + R_2 \leq H(Y|ST) \leq H(Y) \quad (24b)$$

We apply the technique used in [7] to the state-dependent erasure MAC. By focusing on the symmetric rate, we define $p_t \stackrel{\Delta}{=} \Pr(T = t)$, $a_t \stackrel{\Delta}{=} \Pr(X_k = 1|T = t)$ for $k = 1, 2$. By noticing that $H(Y|(s_1, s_2), X_2T)$ is positive only for $(s_1, s_2) = (1, 0), (1, 1)$ and $H(Y|(s_1, s_2), X_1T)$ is positive only for $(s_1, s_2) = (0, 1), (1, 1)$, it readily follows that

$$\begin{aligned} H(Y|SX_2T) = H(Y|SX_1T) &= p_s \sum_t p_t H_2(a_t) \\ &= p_s H_2(\phi(2\sum_t p_t a_t \bar{a}_t)) \end{aligned} \quad (25)$$

$$\begin{aligned}
f_{1a} &= 2\bar{p}_s p_s H_2(r) + p_s^2 H_3(r^2, 2r\bar{r}, \bar{r}^2) \\
f_{1b} &= -\bar{p}\kappa[(\bar{p}_s + p_s\kappa)\log(\bar{p}_s + p_s\kappa) + (p_s\bar{\kappa})\log(p_s\bar{\kappa})] \\
&\quad - p\bar{\kappa}[(\bar{p}_s + p_s\bar{\kappa})\log(\bar{p}_s + p_s\bar{\kappa}) + (p_s\kappa)\log(p_s\kappa)] \\
&\quad - \bar{p}\bar{\kappa}[(\bar{p}_s^2 + p_s\bar{p}_s\kappa)\log(\bar{p}_s^2 + p_s\bar{p}_s\kappa) + (\bar{p}_s p_s + p_s\bar{p}_s\bar{\kappa} + p_s^2\kappa)\log(\bar{p}_s p_s + p_s\bar{p}_s\bar{\kappa} + p_s^2\kappa) + (p_s^2\bar{\kappa})\log(p_s^2\bar{\kappa})] \\
&\quad - p\kappa[(\bar{p}_s^2 + p_s\bar{p}_s\bar{\kappa})\log(\bar{p}_s^2 + p_s\bar{p}_s\bar{\kappa}) + (\bar{p}_s p_s + p_s\bar{p}_s\kappa + p_s^2\bar{\kappa})\log(\bar{p}_s p_s + p_s\bar{p}_s\kappa + p_s^2\bar{\kappa}) + (p_s^2\kappa)\log(p_s^2\kappa)] \\
f_{1c} &= -(\bar{p}\bar{q}\kappa + pq\bar{\kappa})[(\bar{p}_s + p_s\bar{r})\log(\bar{p}_s + p_s\bar{r}) + p_s r\log(p_s r)] \\
&\quad - (\bar{p}q\kappa + p\bar{q}\bar{\kappa})[(\bar{p}_s + p_s r)\log(\bar{p}_s + p_s r) + p_s\bar{r}\log(p_s\bar{r})] \\
&\quad - (\bar{p}\bar{q}\bar{\kappa} + pq\kappa)[(\bar{p}_s^2 + p_s\bar{p}_s\bar{r})\log(\bar{p}_s^2 + p_s\bar{p}_s\bar{r}) + (p_s\bar{p}_s + p_s\bar{p}_s r + p_s^2\bar{r})\log(p_s\bar{p}_s + p_s\bar{p}_s r + \bar{r}) + (p_s^2 r)\log(p_s^2 r)] \\
&\quad - (\bar{p}q\bar{\kappa} + p\bar{q}\kappa)[(\bar{p}_s^2 + p_s\bar{p}_s r)\log(\bar{p}_s^2 + p_s\bar{p}_s r) + (p_s\bar{p}_s + p_s\bar{p}_s\bar{r} + p_s^2 r)\log(p_s\bar{p}_s + p_s\bar{p}_s\bar{r} + p_s^2 r) + (p_s^2\bar{r})\log(p_s^2\bar{r})] \\
f_2 &= 2p_s\bar{p}_s H_2(p\kappa + \bar{p}\bar{\kappa}) + p_s^2 H_3(\bar{p}\kappa^2 + p\bar{\kappa}^2, 2\kappa\bar{\kappa}, p\kappa^2 + \bar{p}\bar{\kappa}^2)
\end{aligned} \quad (21)$$

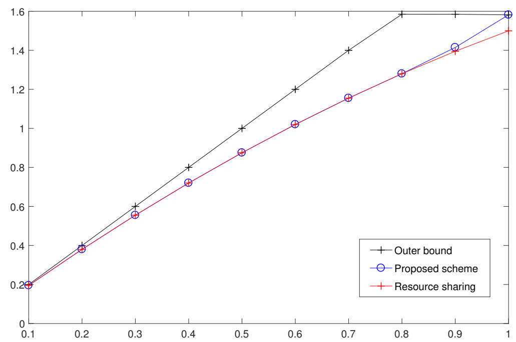

Fig. 2. Unconstrained sum rate vs. state probability p_s.

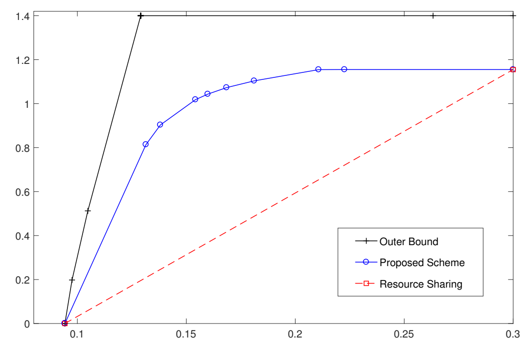

Fig. 3. Tradeoff between sum rate and distortion for $p_s = 0.7$.

where we defined a function $\phi(t) = \frac{1}{2}(1 - \sqrt{1-2t})$ for $t \in [0, 1/2]$ and used the concavity of $H_2(\phi(t))$. We also have

$$\begin{aligned} H(Y) &= H_3(\bar{p}_s^2 + 2p_s\bar{p}_s \sum_t p_t \bar{a}_t + p_s^2 \sum_t p_t \bar{a}_t^2, \\ &\quad 2p_s\bar{p}_s \sum_t p_t a_t + 2p_s^2 \sum_t p_t a_t \bar{a}_t, p_s^2 \sum_t p_t a_t^2) \\ &\le H_2\left(2p_s\bar{p}_s \sum_t p_t a_t + 2p_s^2 \sum_t p_t a_t \bar{a}_t\right) \\ &\quad 1 - [2p_s\bar{p}_s \sum_t p_t a_t + p_s^2 \sum_t p_t (2a_t\bar{a}_t)] \end{aligned} \quad (26)$$

where the last inequality follows from $H_3(a,b,c) = \frac{H_3(a,b,c)+H_3(c,b,a)}{2} \le H_2(b) + 1 - b$, By noticing that the bounds in (25) and (26) depend only on two parameters $\alpha = 2\sum_t p_t a_t \bar{a}_t$ and $\gamma = \sum_t p_t a_t$, we readily obtain

$$\begin{aligned} R_{\text{sum}-\text{out}}(\infty) &= \max_{\alpha,\gamma} \min\{2p_s H_2(\phi(\alpha)), H_2(2p_s\bar{p}_s\gamma + p_s^2\alpha) \\ &\quad + 1 - (2p_s\bar{p}_s\gamma + p_s^2\alpha)\} \\ &= \max_{\beta,\gamma} \min\{2p_s H_2(\beta), H_2(2p_s\bar{p}_s\gamma + 2p_s^2\beta\bar{\beta}) \\ &\quad + 1 - (2p_s\bar{p}_s\gamma + 2p_s^2\beta\bar{\beta})\}. \end{aligned}$$

where the last equality follows by letting $\beta = \phi(\alpha)$, or equivalently $\alpha = 2\beta\bar{\beta}$. The minimum distortion can be calculated similarly to (22) by replacing the estimation cost $\underline{c}_1(x_1, v_2)$ with the idealized estimation cost $c_1(x_1, x_2)$.

D. Numerical Result

Fig. 2 shows the unconstrained sum rate performance as a function of the state probability p_s. For the case of $p_s = 1$, the sum capacity is 1.5822 bit/channel use. The proposed scheme yields a visible gain with respect to the resource-sharing for $p_s > 0.8$ when feedback becomes useful for the unconstrained sum rate. The outer bound is not very tight for p_s closed to one. Fig. 3 shows the tradeoff between the sum rate and the symmetric distortion for $p_s = 0.7$. The proposed scheme achieves a significant gain compared to the resource sharing scheme in terms of tradeoff. Moreover, the proposed scheme achieves near-optimal performance for small distortion values.

Although restricted to a very simple setup, the current work demonstrates a high potential of joint sensing and communication, that exploits feedback both for state sensing and communication.

VI. Acknowledgement

The work of Mari Kobayashi is supported by an Alexander von Humboldt Research Fellowship.

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