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Performance analysis of turbo codes over the Bernoulli-Gaussian impulsive noise channel

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***Abstract*—This paper investigates the performance of turbo codes transmitted over the Bernoulli-Gaussian impulsive noise channel. First, lower bounds on the performance of turbo codes in the error floor region are derived based on the union bound. Second, infinite length analysis of turbo codes is proposed through the method of density evolution that provides a performance measure of these codes in the waterfall region. Finally, bit error rate curves based on Monte Carlo simulations are shown together with the proposed bounds.**

***Index Terms*—Turbo codes, iterative decoding, density evolution, impulse noise, Bernoulli-Gaussian channel.**

I. INTRODUCTION

The impact of impulse noise on communication systems such as in digital subscriber line (DSL) networks [1] and in powerline communication systems [2] has long been studied. Several statistical models have been developed to describe the behavior of impulse noise [3], among which the Bernoulli-Gaussian noise model [4], [5] is of practical interest due to its simplicity in approximating the impulsive behavior for various communication channels.

The design of error correcting codes for impulsive noise channels has been a very active research topic for the past years. Known for their flexibility in terms of coding rates and codeword lengths, trellis codes, whether convolutional or turbo codes, have been the main candidates in both wired and wireless standards. Standards developed for narrowband communication over powerline, such as PRIME (Powerline-Related Intelligent Metering Evolution) [6] and G3 [7], choose convolutional codes as *the* forward error correction method. Some works have recently dealt with the design of convolutional codes for impulsive noise channels in which bits facing a pulse of noise are either brought to a bit level (*i.e.* clipping) or simply set as an erased bit (*i.e.* blanking) [8], [9]. For instance, the authors in [10], [11] propose decoding algorithms for codes that take into account both erasures (due to impulse noise) and noisy observations, and the authors in [12], investigate the performance of convolutional codes over narrowband impulsive noise channels. In [13], the authors provide an analytical framework for the performance of convolutional codes over the Bernoulli-Gaussian noise channel. However, no such analysis exists for turbo codes [14], which are among the best known codes for approaching the capacity of various types of channels. Unlike convolutional codes, the performance of turbo codes is determined by two error rate regions: the waterfall region, due to the interleaving gain of the code, and the error floor region, due to the poor free (minimum) distance of the code [15]. For these reasons, in this paper, we propose a framework for analyzing turbo codes over the Bernoulli-Gaussian channel in the two aforementioned error rate regions.

The rest of this paper is organized as follows: in Section II, we present the system model. In Section III, bounds on the performance of turbo codes in both the error floor (Section III-A) and waterfall regions (Section III-B) are proposed. Section IV presents bit error rate performance results using Monte Carlo simulations, Section V gives the concluding remarks.

II. SYSTEM MODEL

We consider transmission over the Bernoulli - Gaussian impulsive noise channel (BGC), whose model is shown in Fig. 1.

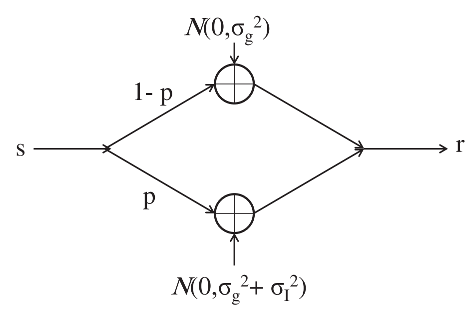

Fig. 1. The Bernoulli Gaussian impulsive noise channel

The received symbol is $r = s+\eta$, where s is the transmitted Binary Phase-Shift Keyed (BPSK) symbol (*i.e.* taking values $+1$ or -1), and η is a Bernoulli-Gaussian complex noise

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realization with probability density function:

$$f = \begin{cases} \mathcal{N}(0, \sigma_g^2), & \text{w.p. } 1-p \\ \mathcal{N}(0, \sigma_g^2 + \sigma_I^2), & \text{w.p. } p \end{cases}$$

where "w.p." means "with probability", p being the probability of occurrence of a noise pulse, σ_g^2 being the variance of the Gaussian noise (denoted as N_0 in the sequel), σ_I^2 being the variance of the impulse noise, and $\mathcal{N}(0, \sigma^2)$ represents a normal (Gaussian) distribution with zero mean and variance σ^2. For the analysis of linear codes over such channels, a symbol corrupted by a noise pulse (with probability p) will be processed before being fed to the iterative receiver in a goal to enhance the performance [8]. We assume perfect detection of the impulse noise, *i.e.* the location of the pulses is known at the receiver. The first method, known as *clipping*, consists of setting the level of a received symbol that crosses a threshold τ to the level of the closest constellation symbol as:

$$y = \begin{cases} r, & \text{if } |r| \leq \tau \\ \gamma, & \text{if } |r| > \tau \end{cases}$$

The second method, called *blanking*, consists of setting the received signal amplitude to zero when it crosses τ.

$$y = \begin{cases} r, & \text{if } |r| \leq \tau \\ 0, & \text{if } |r| > \tau \end{cases}$$

The setup of the threshold value τ has already been studied in the literature: the authors in [8] use numerical optimization to choose the threshold value, while the authors in [9] investigate on the optimization in closed form. It has been extensively shown, namely in [13], that blanking is way superior to clipping over the BGC, as in the case of a noise pulse that has a large amplitude, it is best to provide the decoder with a "neutral" value of 0 (*i.e.* a probability on the bit of 0.5), than to feed it with a biased value. In other words, with BPSK transmission, the channel observation on a coded symbol $x \in \{\pm A\}$ is proportional to $p(y/x) = e^{-(y-x)^2/2\sigma_g^2}/\sqrt{2\pi}\sigma_g$, and can be written as:

$$obs(x) = \begin{cases} \frac{1}{1+\exp\left(-\frac{2Ay}{\sigma_g^2}\right)}, & \text{w.p. } 1-p \\ 0.5, & \text{w.p. } p \end{cases}$$

In the sequel, the encoder is a turbo encoder consisting of two identical Recursive Systematic Convolutional (RSC) encoders in parallel separated by an interleaver of length K [14]. The decoder consists of two soft-input soft-output (SISO) channel decoder that iteratively generate extrinsic probabilities on the bits based on the "Forward-Backward" algorithm [16]. At the last iteration, the decision is taken on the *a posteriori* probability on the bit which is given by the product of the observation, the extrinsic probability computed by the decoder, and the *a priori* probability on the bit. In the next section, bounds on the performance of turbo codes transmitted over the BGC will be derived, both in the error floor and in the waterfall regions.

III. Error bounds on the performance of turbo codes over the Bernoulli-Gaussian channel

The performance of turbo codes [14] over ergodic channels is characterized by two error rate regions: the waterfall region for low signal-to-noise ratios (SNR), and the error floor region for high SNR. The error floor is a result of the poor minimum distance of these codes, while the fast drop in the error rate in the waterfall region is due to the advantageous distance spectrum, *i.e.* to the fact that a small number of codewords are within small distances from each others. The performance of turbo codes in the error floor region has been analyzed through bounds on the the performance for various channels, like for instance for Gaussian noise channels in [15]. On the other hand, the performance in the waterfall region can be linked to the convergence behavior of the iterative decoder, *i.e.* to the performance of turbo codes for large block lengths. This can be assessed through the computation of density functions as they evolve from one iteration to the next. This technique, known as *density evolution*, has been used for other families of capacity-approaching codes, like low density parity-check (LDPC) codes for instance [17]. In this section, we will first propose a lower bound on the minimum distance of turbo codes over the BGC. We will then provide decoding thresholds for these codes over the BGC using the density evolution technique.

A. Lower bound on the performance of turbo codes over the Bernoulli-Gaussian channel

In this section, we will derive a union bound on the performance of turbo codes in the error floor region. In the presence of impulse noise (occurring with probability p), as mentioned in the previous section, the bits that encounter a noise pulse are either clipped to a bit level (either 0 or 1), or blanked (to represent a bit erasure). In both cases, impulse noise will add a penalty in the performance with respect to classical Gaussian noise. For this reason, we consider first $\phi_b\left[\omega(c)\right] \in [0, 1]$, the penalty factor for trellis codes due to blanking [13] for a codeword c of Hamming weight $\omega(c)$.

Proposition 1: Under blanking, the penalty factor introduced by Bernoulli-Gaussian impulsive noise channel on the performance of a convolutional code is given by:

$$\phi_b\left[\omega(c)\right] = \frac{\sum_{i=0}^{\omega(c)} \left(\omega(c) - \frac{i}{2}\right) \binom{\omega(c)}{i} (1-p)^{\omega(c)-i} p^i}{\omega(c)} \quad (1)$$

Proof: We consider Binary Phase-Shift Keying (BPSK) and the 4-state half-rate Recursive Systematic Convolutional (RSC) $(1, 5/7)_8$ code whose trellis is shown in Fig. 2, knowing that the analysis can be generalized to any RSC code and any modulation scheme. The all-zero (blue dashed) codeword and the red dotted codeword differ by $\omega(c) = d_{free} = 5$ bits. In each position in which the two codewords differ, the Euclidean distance generated is $2\sqrt{E_c}$, where $E_c = R_c E_b$. Now suppose that i bits out of the $\omega(c)$ in which the codewords differ are subject to a noise pulse (with probability p) and that they are blanked at the decoder's input. In this case, a bit corresponding to a blanked symbol has a probability of 0.5, which is midway between the two possible values for the bit (*i.e.* a 0 or a 1). This will result in a Euclidean distance of $\sqrt{E_c}$ between the two codewords (*i.e.* half the Euclidean distance generated between 0 and a 1, which is of $2\sqrt{E_c}$). This means that the two codewords will now differ by a Euclidean distance of $\sqrt{E_c}$ (with probability $1-p$) in $\omega(c)-i$ positions, and they will not differ (with probability p) in the remaining i positions. Now if we consider the different values of i between 0 and $\omega(c)$, we obtain a binomial expression as in the numerator of (1). The denominator in the expression of $\phi_b(\omega(c))$ is used for normalizing the expression, so that we obtain $\phi_b(\omega(c)) = 1$ whenever $p = 0$ and hence $i = 0$. □

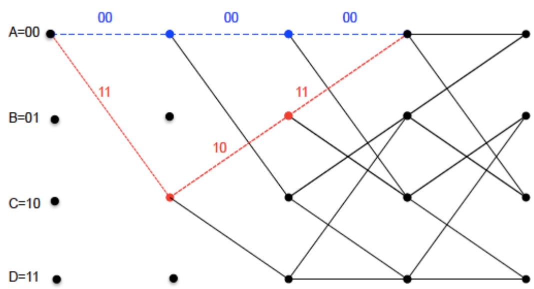

Fig. 2. Trellis of the 4-state half-rate RSC $(1, 5/7)_8$ code

We will now propose a lower bound on the minimum distance of turbo codes on the BGC as a function of $\phi_b[\omega(c)]$ based on results from [15], [18]. As discussed in Section II, only blanking of symbols subject to noise pulses will be considered, as its superiority to clipping applies to any soft-input decoder.

Proposition 2: Under blanking, the union bound on the bit error rate (BER) of a parallel turbo code at a distance $d(j)$ with overall rate R_c is given by:

$$B(u) = \sum_{j=1}^{u} \frac{N_j \omega(j) \phi_b[\omega(j)]}{K} \mathrm{Q}\left(\sqrt{\frac{2E_b}{N_0} d(j) \phi_b[d(j)] R_c}\right) \qquad (2)$$

where $\phi_b[\omega(j)]$ is taken from (1), $d(j)$ is the j^{th} non-zero distance, $\omega(j) = W(j)/N_j$, $W(j)$ is the total information weight of all codewords of weight $d(j)$, and N_j is the number of codewords of weight $d(j)$, and $Q(x) = \frac{1}{2}\mathrm{erfc}(\frac{1}{\sqrt{2}})$, "erfc" being the complementary error function.

Proof: The expression in (2) is based on the union bound on the performance of parallel concatenated turbo codes over AWGN channels with a given distance spectrum [15], [18]. The multiplicative factor depends on the multiplicity of the turbo code in consideration (*i.e.* N_j) and on $\omega(j)$, while the argument of the Q function depends the coding rate R_c and the weight of codeword j, namely $d(j)$. For this reason, at a given codeword j, we assume a penalty factor whose expression is given in (1) affecting both $\omega(j)$ in the multiplicative factor of the union bound and $d(j)$ within the Q function. □

The union bound described in (2) gives a lower bound on the performance of a turbo code while considering u codewords from the distance spectrum. All the parameters of the expression (*i.e.* N_j, $\omega(j)$, and $d(j)$) can be obtained through well-known algorithms [15], [18]. Some examples will be considered in the sequel for Monte-Carlo simulations. We will next propose a way to compute the infinite length performance bound of turbo code, which corresponds to the "waterfall region" in the error rate curves, based on the method of probability density evolution, or simply "density evolution" [17].

B. Infinite length performance of turbo codes over the Bernoulli-Gaussian channel: Density evolution

The density evolution (DE) method is widely known by researchers in the coding community for giving the limiting convergence behavior of capacity-approaching codes for large block lengths [17]. It has been used to find optimal degree profiles for irregular Low density Parity-Check (LDPC) codes in [17], [19], [20] or even irregular turbo codes [21], [22]. In this paper, we propose to use this method to determine the performance limits of turbo codes transmitted over the BGC for large block lengths as a function of the probability of noise impulses p. To start with, we consider rate-R_c turbo codes built from two rate-ρ constituent RSC codes. Due to the symmetry of the channel, we assume that the all-zero codeword is modulated into $\boldsymbol{x} = -1, -1, ..., -1$ and transmitted over the Bernoulli-Gaussian noise channel of Fig. 1. At the channel output, and with blanking, each received sample can be written as:

$$y = \begin{cases} x + \eta = -1 + \eta, & \text{w. p. } 1-p \\ 0.0, & \text{w. p. } p \end{cases}$$

This implies that the log-likelihood ratio (LLR) is given by the well-known expression:

$$\mathcal{M}_0 = \begin{cases} \log \frac{p(y|x=-1)}{p(y|x=+1)} = \frac{2}{N_0} y = -\frac{2}{N_0}(1+\eta), & \text{w. p. } 1-p \\ 0, & \text{w. p. } p \end{cases} \qquad (3)$$

We have that $\mathcal{M}_0 \sim (1-p)\mathcal{N}(-\frac{2}{N_0}, \frac{4}{N_0}) + p.0$, and thus the associated probability density function will be denoted by $p_0(x)$.

The local neighborhood tree for an information bit belonging to an acyclic asymptotically large turbo code is shown in Fig. 3.

The index i refers to the decoding iteration number. A bitnode has one incoming extrinsic probability ξ_i and one outgoing *a priori* probability π_i which also plays the role of a partial *a posteriori* probability (APP). The total APP may be obtained by combining π_i with an extra extrinsic probability. The message associated to ξ_i is $\mathcal{M}_i = \frac{\log(\xi_i(\text{bit}=0))}{\log(\xi_i(\text{bit}=1))}$ and its probability density function is $p_{\mathcal{M}_i}(x)$. The probability density function of log-ratio messages associated to π_i will be denoted by $p_i(x)$. Following [17] we have that

$$p_i(x) = \mathcal{F}^{-1}\left[\mathcal{F}[p_0(x)]\,\mathcal{F}[p_{\mathcal{M}_i}(x)]\right] \qquad (4)$$

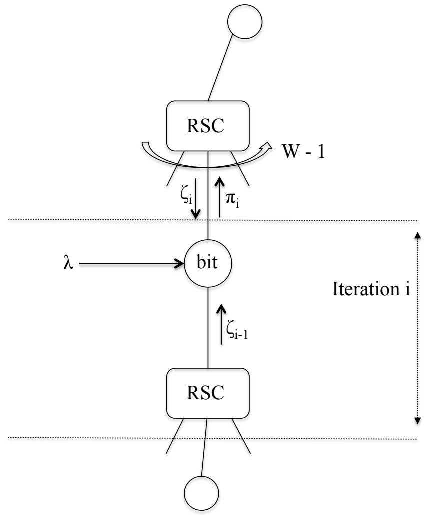

Fig. 3. Propagation tree used in density evolution for an irregular turbo code. The π_i represents *a priori* probability, and the ξ_i the extrinsic probability. Circles represent bitnodes, and rectangles are local neighborhood RSC trellis constraints.

where $\mathcal{F}$ denotes the Fourier transform operator. Based on partial a posteriori probabilities, the average bit error probability at iteration i is defined as

$$P_b(i) = \int_{-\infty}^{0} p_i(x)dx \tag{5}$$

At an RSC checknode level, and as illustrated in Fig. 3, based on a priori input π_{i-1} with probability density function (PDF) $p_{i-1}(x)$, an accurate estimation of $p_{\mathcal{M}_i}(x)$ is made via a forward-backward algorithm [16] applied on a sufficiently large trellis window of size W centered around the information bit. Given a turbo code ensemble, the decoding *threshold* is the minimal signal-to-noise ratio E_b/N_0 for which $P_b(i)$ vanishes with i. The threshold can be determined via Density Evolution (DE) [17], a procedure where $p_i(x)$ is updated from $p_{i-1}(x)$ by propagating probabilistic densities through the tree graph of Fig. 3.

C. Numerical results for AWGN

As an example, for a rate $R_c = 1/3$ parallel turbo code built from two half-rate RSC $(13, 15)_8$ constituent codes, the convergence threshold for $p = 0.01$ was of -0.03 dB, while the threshold for $p = 0.1$ was of 0.53 dB. Similarly, for a half-rate parallel turbo code built from two half-rate RSC $(37, 21)_8$ constituent codes with puncturing, the convergence threshold for $p = 0.01$ was of 0.57 dB, while the threshold for $p = 0.1$ was of 1.25 dB.

IV. Simulation Results

In this section, bit error rate performances of parallel turbo codes over the BGC with blanking under iterative decoding are shown using Monte Carlo simulations. The modulation is BPSK, and the probability of a noise pulse is $p = 0.1$. Error rate curves are compared to union bounds and density evolution thresholds derived in Section III. Fig. 4 shows the performance of the parallel $R_c = 1/3$ turbo code obtained from two half-rate 8-state RSC $(1, 13/15)_8$ constituent codes. For $K = 1000$, a Progressive-Edge Growth (PEG) interleaver is used [23], which ensures that the free distance of the code increases as $\log(K)$ (*i.e.* optimal growth). At this interleaver size, the distance spectrum of the code is given in Table I below.

TABLE I
Distance spectrum of the $R_c = 1/3$ turbo code, RSC $(1, 13/15)_8$ constituent codes, $K = 1000$ from [18]

j	$d(j)$	$N(j)$	$\omega(j)$
1 (d_{free})	21	1	2
2	22	6	72
3	24	2	6
4	25	2	12

As $d(2)$ is very close to $d(1) = d_{free}$, a better estimation of the error floor bound is obtained by considering $B(2)$ instead of $B(1) = B(d_{free})$. However, taking more terms in the union bound does not improve the bound. On the other hand, the density evolution threshold is a good approximation for the code performance but for a large interleaver size (namely $K = 100000$), where pseudo-random interleaving is used. Fig. 5 shows the performance of the parallel $R_c = 1/2$ turbo code obtained from two half-rate 64-state RSC $(1, 21/37)_8$ constituent codes with puncturing. In this case, the distance spectrum for $K = 65536$ is given in Table II below.

TABLE II
Distance spectrum of the $R_c = 1/2$ turbo code, RSC $(1, 21/37)_8$ constituent codes, $K = 65536$ from [15]

j	$d(j)$	$N(j)$	$\omega(j)$
1 (d_{free})	6	4.5	9
2	8	11	22
3	10	20.5	41
4	12	75	150

Now as $d(2)$ is larger than d_{free}, there is no significant improvement in the error floor bound by moving from $B(1) = B(d_{free})$ to $B(2)$. As for the waterfall region, the density evolution threshold gives a good estimate of the code performance.

V. Conclusion

We proposed bounds on the bit error rate performance of turbo codes over the Bernoulli-Gaussian impulsive noise channel with blanking under iterative decoding. In the error floor region, a union bound on the bit error rate was derived. In the waterfall region, an infinite length analysis was provided based on probability density evolution that gives a bound on the performance of the code. Bit error rate results obtained from Monte Carlo simulations were shown to fit with the proposed bounds.

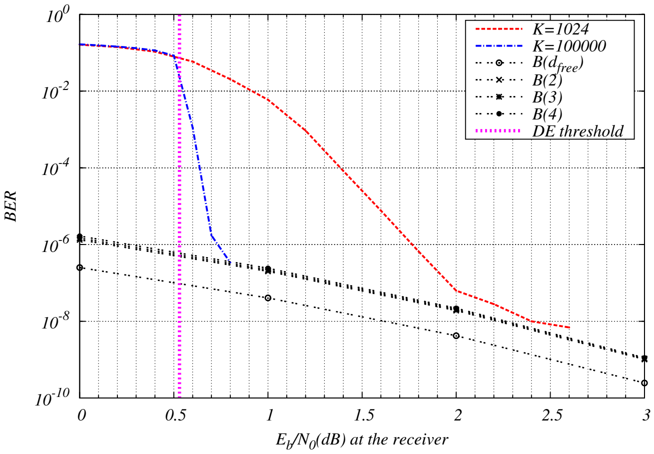

Fig. 4. Performance of a parallel rate-$1/3$ turbo code over the BGC with blanking, $p = 0.1$, BPSK modulation, half-rate 8-state RSC $(1, 13/15)_8$ constituent codes, PEG interleaving, with lower union bound and density evolution threshold.

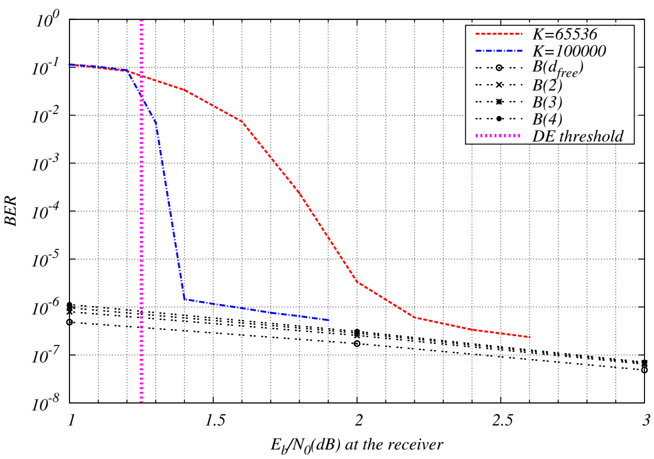

Fig. 5. Performance of parallel half-rate turbo codes over the BGC with blanking, $p = 0.1$, BPSK modulation, half-rate 64-state RSC $(1, 21/37)_8$ constituent codes, pseudo-random interleaving, with lower union bound and density evolution threshold.

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