# Performance Analysis of Convolutional Codes over the Bernoulli-Gaussian Impulsive Noise Channel

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***Abstract*—This paper investigates the performance of convolutional codes with quadrature amplitude modulation transmitted over the Bernoulli-Gaussian impulsive noise channel. First, the performance superiority of blanking over clipping of the symbols affected by noise pulses is proved through the computation of a lower bound on the bit error rate. Next, lower and upper bounds on the bit error rate performance are derived. Finally, bit error rate curves based on Monte Carlo simulations are shown together with the proposed bounds.**

***Index Terms*—Coded modulations, iterative decoding, convolutional codes, impulse noise, Bernoulli-Gaussian channel.**

## I. Introduction

IMPULSE noise is known to be present in many communication systems, such as in digital subscriber line (DSL) networks [1] and in powerline communication systems [2]. Several statistical models have been developed to describe the behavior of impulse noise [3], that are the Bernoulli-Gaussian model [4], [5], the Gaussian-Mixture model [6], and the Middleton Class-A, Class-B, and Class-C [7]. Among these models, the Bernoulli-Gaussian noise model is of practical interest due to its simplicity in approximating the impulsive behavior for various communication channels [5], [8], [9].

The design of error correcting codes for impulsive noise channels has been a very active research topic for the past years. Known for their flexibility in terms of coding rates and codeword lengths, convolutional codes have been the main candidates in both wired and wireless standards. Standards developed for narrowband communication over powerline, such as PRIME (Powerline-Related Intelligent Metering Evolution) [10] and G3 [11], choose convolutional codes as *the* forward error correction method. Some works have recently dealt with the design of convolutional codes for impulsive noise channels in which bits facing an pulse of noise are either brought to a bit level (*i.e.* clipping) or simply set as an erased bit (*i.e.* blanking) [12]. For instance, the authors in [14], [15] propose decoding algorithms for codes that take into account both erasures (due to impulse noise) and noisy observations, however only hard decision decoding is considered. In [16], the authors investigate the performance of convolutional codes over narrowband impulsive noise channels, however the work does not provide theoretical analysis of the performance. For this reason, in this paper, we propose a framework for analyzing convolutional coded modulations over the Bernoulli-Gaussian channel under both clipping and blanking.

The rest of this paper is organized as follows: in Section II, we present the system model. In Section III, a performance comparison of coded modulations under both clipping and blanking is shown, which leads to bounds on the error rate performances of such schemes. Section IV presents bit error rate performance results using Monte Carlo simulations, Section V gives the concluding remarks.

## II. System Model

We consider transmission over the Bernoulli - Gaussian channel, whose model is shown in Fig. 1. The channel model is $r = s + n$, where $s$ is an $M = 2^m$-Quadrature Amplitude Modulated (QAM) symbol, where $m$ is the number of bits per symbol, and $n = n_I + jn_Q$ is a Bernoulli-Gaussian complex noise realization with probability density function:

$$F(n_I) = F(n_Q) = (1-p)\mathcal{N}(0, \sigma_g^2) + p\mathcal{N}(0, \sigma_g^2 + \sigma_I^2)$$

where $p$ is the probability of occurrence of a noise pulse, $\sigma_g^2$ is the variance of the Gaussian noise (denoted as $N_0$ in the sequel), $\sigma_I^2$ is the variance of the impulse noise, and $\mathcal{N}(0, \sigma^2)$ represents a normal (Gaussian) distribution with zero mean and variance $\sigma^2$.

For the analysis of linear codes over such channels, a symbol corrupted by a noise pulse (with probability $p$) will be processed before being fed to the iterative receiver in a goal to enhance the performance [12]. We assume perfect detection of the impulse noise, *i.e.* the location of the pulses is known at the receiver. The first method, known as *clipping*, consists of setting the level of a received symbol that crosses a threshold $\tau$ to the level of the closest constellation symbol $\gamma \in M$-QAM, as:

$$y = \begin{cases} r, & \text{if } |r| \leq \tau \\ \gamma, & \text{if } |r| > \tau \end{cases}$$

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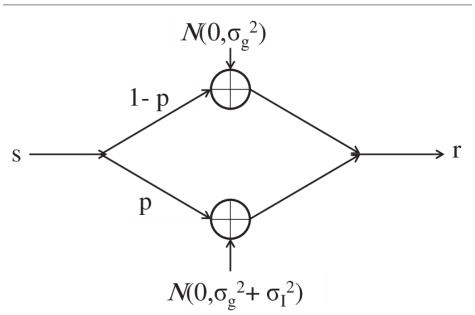


Fig. 1. The Bernoulli Gaussian impulsive noise channel

The second method, called *blanking*, consists of setting the received signal amplitude to zero when it crosses $\tau$.

$$y = \begin{cases} r, & \text{if } |r| \leq \tau \\ 0, & \text{if } |r| > \tau \end{cases}$$

The setup of the threshold value $\tau$ has already been studied in the literature: the authors in [12] use numerical optimization to choose the threshold value, while the authors in [13] investigate on the optimization in closed form. In Section III, the effect of the two schemes on the performance of linear codes will be considered.

In addition to the above described channel model, digital transmission is performed as follows: an information packet of length $K$ bits is fed to a recursive systematic convolutional (RSC) code that generates $P$ parity bits. The codeword of length $N = K + P$ bits is then fed to an interleaver before being modulated and sent over the channel. The receiver consists of a soft-input soft-output (SISO) detector that generates extrinsic probabilities $\xi(c_i)$ on received bits based on symbol $y$ and *a priori* probabilities $\pi(c_u)$ as:

$$\begin{aligned} \xi(c_i) &= \frac{p(y/c_i = 1)}{p(y/c_i = 0) + p(y/c_i = 1)} \\ &= \frac{\sum_{s' \in M(c_i=1)} \left[ \left( e^{-\|y-s'\|^2/2N_0} \right) \prod_{u \neq i} \pi(c_u) \right]}{\sum_{s \in M} \left[ \left( e^{-\|y-s\|^2/2N_0} \right) \prod_{u \neq i} \pi(c_u) \right]} \end{aligned}$$

The soft values of the bits (corresponding to extrinsic probabilities) are then deinterleaved before being fed to a soft-input soft-output "Forward-Backward" decoder [17] that generates extrinsic probabilities on the bits. The resulting soft values of the bits are then fed to an interleaver and back to the detector. After few iterations, a decision on the information bits is made at the output of the decoder based on their *a posteriori* probabilities (APP), as shown in Fig. 2.

## III. Error Bounds for Coded Modulations

The performance of convolutional codes over Gaussian noise channels has long been studied, and bounds on the performance of such codes have been derived [18]. For uncoded $M$-QAM transmission, the bit error rate (BER) is given by:

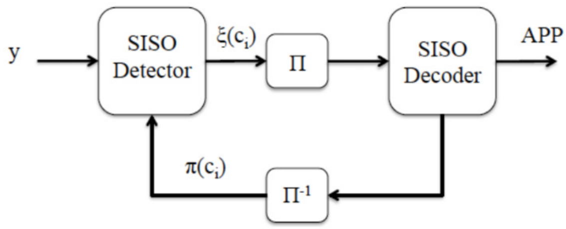


Fig. 2. Iterative soft demodulator/decoder for a convolutional-coded modulation.

$$P_{eb} = 2\left(\frac{\sqrt{M}-1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3m}{M-1}}\sqrt{\frac{E_b}{N_0}}\right)$$

In the presence of a convolutional code with rate $R_c$, and assuming the all zero codeword has been transmitted, the pairwise error probability (PEP) with $M$-QAM is given by:

$$P_{ep}[\omega_H(c)] = P(0 \to c) = 2\left(\frac{\sqrt{M}-1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3m}{M-1}}\sqrt{\frac{R_c \omega_H(c) E_b}{N_0}}\right)$$

where $\omega_H(c)$ is the Hamming weight of codeword $c$.

### A. Penalty factor due to impulsive noise

In the presence of impulse noise, as mentioned in the previous section, the bits that encounter a noise pulse are either clipped to a bit level (either 0 or 1), or blanked (to represent a bit erasure). In both cases, impulse noise will introduce a penalty in the performance as compared to pure Gaussian noise. For this reason, we introduce $\phi_c[\omega_H(c)] \in [0, 1]$ and $\phi_b[\omega_H(c)] \in [0, 1]$, the penalty factors due to clipping and blanking respectively.

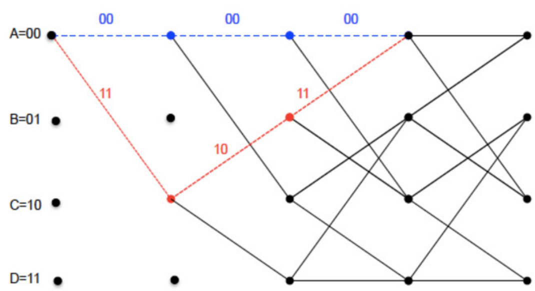


Fig. 3. Trellis of the 4-state half-rate RSC $(1, 5/7)_8$ code

*Proposition 1:* Under *clipping*, the pairwise error probability of a coded modulation transmitted over the Bernoulli-Gaussian impulsive noise channel with convolutional coding and $M$-QAM modulation is:

$$P_{ep,c}[\omega_H(c)] = 2\left(\frac{\sqrt{M}-1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3m}{M-1}}\sqrt{\frac{R_c\omega_H(c)E_b}{N_0}\phi_c\left[\omega_H(c)\right]}\right)$$

where

$$\begin{aligned}\phi_c\left[\omega_H(c)\right] &= \frac{\sum_{i=0}^{\omega_H(c)}\left(\omega_H(c)-i\right)\binom{\omega_H(c)}{i}(1-p)^{\omega_H(c)-i}p^i}{\omega_H(c)} \quad (1)\\ &= 1-p\end{aligned}$$

*Proof*: We consider Binary Phase-Shift Keying (BPSK) and the 4-state half-rate Recursive Systematic Convolutional (RSC) $(1, 5/7)_8$ code whose trellis is shown in Fig. 3, knowing that the analysis can be generalized to any RSC code and any modulation scheme. The all-zero (blue dashed) codeword and the red dotted codeword differ by $\omega_H(c) = d_{free} = 5$ bits. In each position in which the two codewords differ, the Euclidean distance generated is $2\sqrt{E_c}$, where $E_c = R_cE_b$. Now suppose that $i$ bits out of the $\omega_H(c)$ in which the codewords differ are subject to a noise pulse (with probability $p$) and that they are clipped at the decoder's input. In this case, a bit corresponding to a clipped symbol can, in the worst case, result in a Euclidean distance of zero between the two codewords; this is due to the fact that by clipping the amplitude of a modulated symbol, it can be easily confused with another symbol from the constellation. In BPSK for instance, where a level of $-1$ V corresponds to a bit of 0, and a $+1$ V to a bit of 1, if a zero is transmitted and it is subject to a high positive noise pulse, it will take a positive value (close to $+1$ V) after clipping. This means that the two codewords will now differ by a Euclidean distance of $2\sqrt{E_c}$ (with probability $1-p$) in $\omega_H(c)-i$ positions, and they will not differ (with probability $p$) in the remaining $i$ positions, which gives rise to (1) involving a binomial expression. The denominator in the expression of $\phi_c\left(\omega_H(c)\right)$ is used for normalizing the expression, so that we obtain $\phi_c\left(\omega_H(c)\right) = 1$ whenever $p = 0$ and hence $i = 0$. Now if we define $k = \omega_H(c) - i$, (1) can be rewritten as:

$$\phi_c\left[\omega_H(c)\right] = \frac{\sum_{k=0}^{\omega_H(c)} k\binom{\omega_H(c)}{k}(1-p)^k p^{\omega_H(c)-k}}{\omega_H(c)}$$

which reduces to $\phi_c\left[\omega_H(c)\right] = (1-p)\,\omega_H(c)/\omega_H(c) = 1-p$

*Proposition 2:* Under *blanking*, the pairwise error probability of a coded modulation transmitted over the Bernoulli-Gaussian impulsive noise channel with convolutional coding and $M$-QAM modulation is:

$$P_{ep,b}[\omega_H(c)] = 2\left(\frac{\sqrt{M}-1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3m}{M-1}}\sqrt{\frac{R_c\omega_H(c)E_b}{N_0}\phi_b\left[\omega_H(c)\right]}\right)$$

where

$$\phi_b\left[\omega_H(c)\right] = \frac{\sum_{i=0}^{\omega_H(c)}\left(\omega_H(c)-\frac{i}{2}\right)\binom{\omega_H(c)}{i}(1-p)^{\omega_H(c)-i}p^i}{\omega_H(c)} \qquad (2)$$

*Proof*: The proof with blanking is somehow similar to that with clipping, with one major difference: whenever a symbol is blanked, its amplitude is set to zero. Consider for instance BPSK, where a symbol $s_i$ can be written as a function of bit $b_i$ as: $s_i = 2b_i - 1$. If $s_i = 0$ (*i.e.* blanked), this results in $b_i = 0.5$. This means that the corresponding bits take a (soft) value of 0.5, the Euclidean distance between the two codewords at this position will be equal to $\sqrt{E_c}$, and not zero as with clipping. This means that the two codewords will now differ by a Euclidean distance of $2\sqrt{E_c}$ (with probability $1-p$) in $\omega_H(c) - i$ positions, and they will differ by a Euclidean distance of $\sqrt{E_c}$ (with probability $p$) in the remaining $i$ positions, which leads to (2).

### B. Lower bound with clipping and blanking

We will now investigate on the impact of clipping and nulling on the BER performance of convolutional codes. It is known that the dominant PEP in the total error probability is the one related to the codeword having $\omega_H(c) = d_{free}$, the free distance of the convolutional code [18]. This means that one can define a lower bound on the BER under both clipping and blanking as:

$$P_{LB,b,c} = A(d_{free})F(d_{free})P_{ep,b,c}[d_{free}]$$

where $A(d_{free})$ represents the number of codewords at distance $d_{free}$ and $F(d_{free})$ represents the number of bit errors involved in the corresponding error event [18]. The performance comparison between clipping and blanking is highlighted in Fig. 4 below for the half-rate RSC $(1, 5/7)_8$ code at $E_b/N_0 = 7$ dB. This figure shows the superiority of the blanking method, as, in terms of decoding, if a bit is subject to a high noise pulse, the best thing is to deliver a neutral observation on the bit to the decoder (*i.e.* 0.5), rather than "forcing" it to a 0 or a 1.

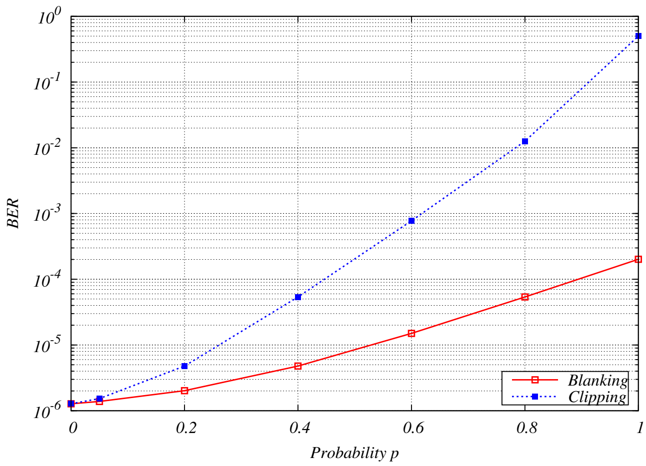


Fig. 4. Comparison between clipping and blanking for the half-rate RSC $(1, 5/7)_8$ code with $d_{free} = 5$ at $E_b/N_0 = 7$ dB.

### C. *Upper bound with blanking*

We again consider the example of the four-state half-rate RSC $(1, 5/7)_8$ code with BPSK modulation. In order to derive the upper bound on the performance, we use the method of the transfer function of the convolutional code [18]. The transfer function gives information about the paths that start from the all-zero state and return to this state for the first time. To obtain the transfer function, the all-zero state is split into two states, one denoting the starting state and the other denoting the first return. To every branch connecting two states corresponds a function of the form $D^\alpha N^\beta J$ where $\alpha$ denotes the number of ones in the output bit sequence (equivalently the Hamming weight of the codeword), $\beta$ denotes the number of ones in the input sequence and the exponent of J will be the number of branches spanned when deriving the final transfer function. The flow graph of the RSC $(1, 5/7)_8$ code is shown in Fig. 5 below:

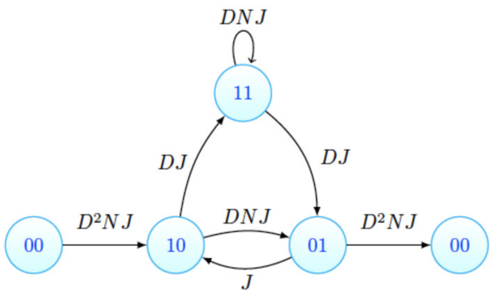


Fig. 5. Flow graph of the half-rate RSC $(1, 5/7)_8$ code.

The transfer function is then the transfer function of the flow graph between the starting all-zero state and the final all-zero state denoted by $T(D, N, J)$. Naming the states as follows: $A = 00$, $B = 01$, $C = 10$, $D = 11$, and $E = 00$ is the final all-zero state. The transfer function is then derived as:

$$T(D, N, J) = \frac{X_E}{X_A}$$

$$= \frac{D^5N^3J^3 - D^6N^4J^4 + D^6N^2J^4}{1 - DNJ - DNJ^2 + D^2N^2J^3 - D^2J^3}$$

In polynomial form, we obtain:

$$T(D, N, J) = D^5N^3J^3 + D^6N^2J^4 + ...$$

where the first term $D^5N^3J^3$ indicates that there is a path starting from the all-zero state and returning to it for the first time, for which the code word has a Hamming weight of $\omega_H(c) = d_{free} = 5$, the input sequence contains three 1's, and it spans three branches. Because we only care about the weight of the code, we can set $T_2(D, N) = T(D, N, J)|_{J=1}$. The upper bound can then be calculated using :

$$P_b \leq \frac{1}{2}\frac{\partial T_2(D, N)}{\partial N}\Big|_{N=1, D=e^{-\phi_b\left[d_{free}\right]\frac{R_cE_b}{N_0}}} \tag{3}$$

Finally, assuming $M$-QAM modulation and soft decision decoding, and using (3), the bound on the probability of bit error for the RSC $(1, 5/7)_8$ code is computed as:

$$P_{upper} = \frac{1}{2}\frac{3D^5 - 6D^6 + 2D^7}{(1 - 2D)^2}\Big|_{D=e^{-\phi_b\left[d_{free}\right]\frac{3m}{M-1}\frac{R_cE_b}{N_0}}}$$

By using the same method, the upper bound on the performance of the half-rate 8-state RSC $(1, 17/15)_8$ code with $M$-QAM is given in (4).

## IV. Simulation results

In this section, bit error rate performances of convolutional code-based coded modulation are shown using Monte Carlo simulations. Error rate curves are compared to lower and upper bounds derived in Section III. Moreover, for large SNR values, the presence of impulse noise makes the receiver face an error floor, as the decoder cannot recover from the erasures caused by noise pulses. This error floor can be exactly computed, as shown in [19]. For this reason, a new bound, called "erasure bound", was introduced in the figures. The erasure bounds for the codes simulated in this Section are taken from [19], [20]. Fig. 6 and 7 shows the performance for half-rate 4-state codes with BPSK and 16-QAM respectively, while Fig. 8 shows the performance for a half-rate 8-state codes with QPSK. For 16-QAM, $s$-random interleaving [21] with $s = 18$ was used to avoid having the four bits of a blanked symbol adjacent in the decoder, thus ensuring statistical independence of the bits at the receiver side. Pseudo-random interleaving is used in the other figures. In the three figures, the simulated results fit within fractions of a $dB$ of the bounds derived in this paper.

## V. Conclusion

We proposed bounds on the bit error rate performance of convolutional codes with quadrature amplitude modulation of any constellation size transmitted over the Bernoulli-Gaussian impulsive noise channel under iterative detection and decoding under. By considering both clipping and blanking of received symbols, we have theoretically proved the superiority of the latter. Bit error rate results obtained from Monte Carlo simulations were shown to fit with the proposed bounds.

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$$P_{upper} = \frac{\sqrt{M}-1}{\sqrt{M}} \left[ \frac{\left(-D^8+D^7+D^6\right)\left(2D^7-2D^5-3D^3+2D\right)}{\left(D^3+2D-1\right)^2} - \frac{\left(2D^{11}-4D^8-D^7+4D^6\right)}{D^3+2D-1} \right] \Big|_{D=e^{-\phi_b[d_{free}]\frac{3m}{M-1}\frac{R_c E_b}{N_0}}} \quad (4)$$

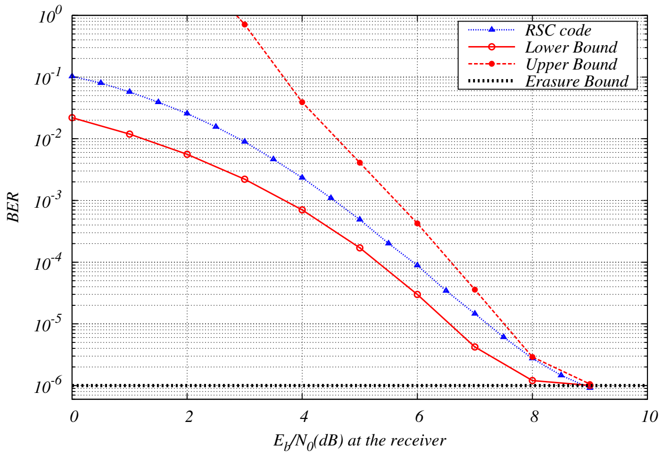


Fig. 6. Performance of convolutional coded BPSK with blanking, half-rate 4-state RSC $(1,5/7)_8$ code, pseudo-random interleaving, $p = 0.05$, $K = 1000$, with lower/upper bound and erasure bound.

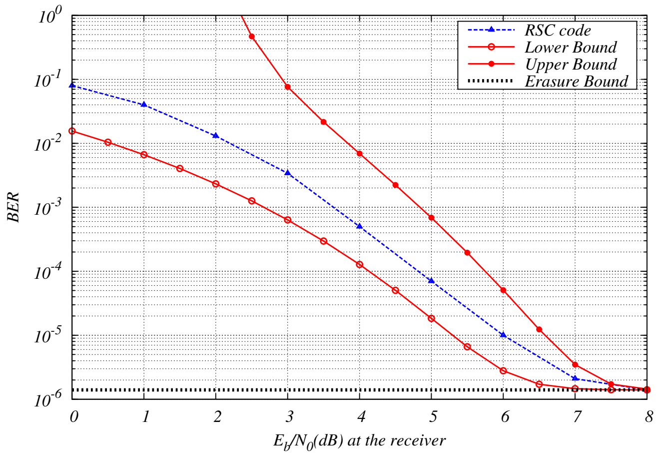


Fig. 8. Performance of convolutional coded BPSK with blanking, half-rate 8-state RSC $(1,17/15)_8$ code, pseudo-random interleaving, 3 receiver iterations, $p = 0.05$, $K = 5000$, with lower/upper bound and erasure bound.

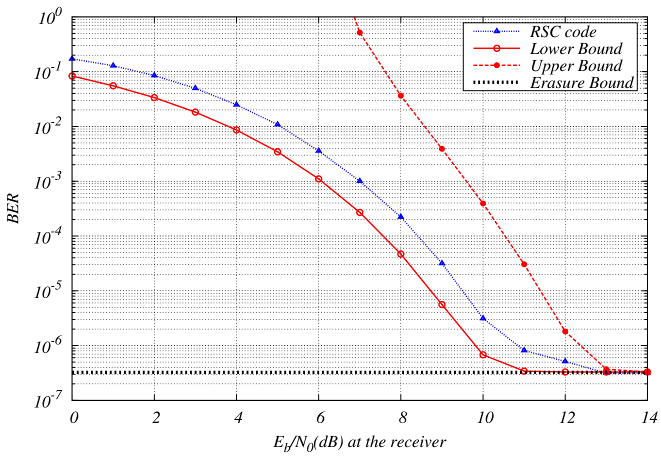


Fig. 7. Performance of convolutional coded 16-QAM with blanking, half-rate 4-state RSC $(1,5/7)_8$ code, s-random interleaving ($s = 18$), 3 receiver iterations, $p = 0.01$, $K = 1000$, with lower/upper bound and erasure bound.

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